

CONTINUED FRACTION NORMALITY IS NOT PRESERVED ALONG ARITHMETIC PROGRESSIONS

BYRON HEERSINK AND JOSEPH VANDEHEY

ABSTRACT. It is well known that if $0.a_1a_2a_3\dots$ is the base- b expansion of a number normal to base- b , then the numbers $0.a_ka_{m+k}a_{2m+k}\dots$ for $m \geq 2$, $k \geq 1$ are all normal to base- b as well.

In contrast, given a continued fraction expansion $\langle a_1, a_2, a_3, \dots \rangle$ that is normal (now with respect to the continued fraction expansion), we show that for any integers $m \geq 2$, $k \geq 1$, the continued fraction $\langle a_k, a_{m+k}, a_{2m+k}, a_{3m+k}, \dots \rangle$ will never be normal.

1. INTRODUCTION

A number $x \in [0, 1)$ with base 10 expansion $x = 0.a_1a_2a_3\dots$ is said to be normal (to base 10) if for any finite string $s = [c_1, c_2, \dots, c_k]$ of digits in $\{0, \dots, 9\}$, we have that

$$\lim_{n \rightarrow \infty} \frac{\#\{0 \leq i \leq n : a_{i+j} = c_j, 1 \leq j \leq k\}}{n} = \frac{1}{10^k}.$$

Although almost all real numbers are normal, we still do not know of a single commonly used mathematical constant, such as π , e , or $\sqrt{2}$, that is normal.

A classical result due to Wall [11] says that if $0.a_1a_2a_3\dots$ is normal, then so is $0.a_ka_{m+k}a_{2m+k}a_{3m+k}\dots$, for any positive integers k, m . In concise terms, sampling along an arithmetic progression of digits preserves normality for base 10 (and more generally, base b) expansions. Sampling along other sequences has been studied most notably by Agafonov [1], Kamae [7], and Kamae and Weiss [8]. Merkle and Reimann [9] studied methods of sampling that do not preserve normality.

However, these works have focused primarily on base- b expansions and so equivalent questions for other expansions are mostly unknown.

In this paper, we consider continued fraction expansions given by

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} = \langle a_1, a_2, a_3, \dots \rangle, \quad a_i \in \mathbb{N}$$

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Email: heersin2@illinois.edu

University of Illinois at Urbana-Champaign

Email: vandehey@uga.edu

University of Georgia.

for $x \in [0, 1)$. The Gauss map T is given by $Tx = x^{-1} - \lfloor x^{-1} \rfloor$ or, if $x = 0$, then $Tx = 0$, and it acts as a forward shift on continued fraction expansions, so that

$$T\langle a_1, a_2, a_3, \dots \rangle = \langle a_2, a_3, a_4, \dots \rangle.$$

The Gauss measure μ on $[0, 1)$ is given by

$$\mu(A) = \int_A \frac{1}{(1+x) \log 2} dx.$$

Given a finite string $s = [d_1, d_2, \dots, d_k]$ of positive integers we define the cylinder set C_s to be the set of points $x \in [0, 1)$ such that the string $[a_1, a_2, \dots, a_k]$ of the first k digits of x equals s . (The expansions of rational numbers are finite and non-unique, but we may ignore such points throughout this paper.)

We say that $x \in [0, 1)$ is CF-normal if, for any finite, non-empty string $s = [d_1, d_2, \dots, d_k]$ of positive integers, we have

$$\lim_{n \rightarrow \infty} \frac{\#\{0 \leq i \leq n : T^i x \in C_s\}}{n} = \mu(C_s),$$

which is equivalent to saying that the limiting frequency of s in the expansion of x equals $\mu(C_s)$, since $T^i x \in C_s$ if and only if the string $[a_{i+1}, a_{i+2}, \dots, a_{i+k}]$ equals s . By the ergodicity of the Gauss map T and the pointwise ergodic theorem, almost all $x \in [0, 1)$ are CF-normal.

Theorem 1.1. *Suppose $\langle a_1, a_2, a_3, \dots \rangle$ is CF-normal. Then the number $\langle a_k, a_{m+k}, a_{2m+k}, a_{3m+k}, \dots \rangle$ is not CF-normal for any integers $k \geq 1$, $m \geq 2$. In fact, for any integers $k \geq 1$, $m \geq 2$, we have that*

$$\lim_{n \rightarrow \infty} \frac{\#\{1 \leq i \leq n : a_{(i-1)m+k} = a_{im+k} = 1\}}{n}$$

exists, but does not equal $\mu(C_{[1,1]})$, so that the CF-normality of $\langle a_k, a_{m+k}, a_{2m+k}, a_{3m+k}, \dots \rangle$ can be seen to fail just by examining the frequency of the string $[1, 1]$.

One of the key techniques in proving this result is a way of augmenting the usual Gauss map T to simultaneously act on a finite-state automata. A number of recent results have made use of this blending of ergodicity and automata. It was used in Agafonov's earlier cited result [1]. It was used in Jager and Liardet's proof of Moeckel's theorem (where it was called a skew product) [6]. It was used to study normality from the viewpoint of compressability [2, 3]. And it was used by Blanchard, Dumont, and Thomas to give reproofs of some classical normality equivalencies, even extending some of these results to what they call "near-normal" numbers [4, 5].

We end the introduction with two questions.

First, the proof of Theorem 1.1 could be extended to show that any continued fraction expansion formed by selecting along a non-trivial arithmetic progression of digits from a CF-normal number has all its 1-digit strings appearing with the right frequency, but the 2-digit string $[1, 1]$ does not. We wonder whether any string with more than one digit can appear with the correct frequency for CF-normality, or are they always incorrect.

Second, as stated earlier, sampling along a non-trivial arithmetic progression preserves normality for base- b expansions. It can be shown, using, say, the augmented systems in this

paper, that a similar result holds for any fibred system that is Bernoulli. The continued fraction expansion is a simple example of a non-Bernoulli system. Is Bernoullicity not only sufficient but necessary for selection along non-trivial arithmetic progressions to preserve normality?

2. AN AUGMENTED SYSTEM

We will require a result from a previous paper of the second author [10].

Let T be the Gauss map acting on the set $\Omega \subset [0, 1)$ of irrationals. So $Tx \equiv 1/x \pmod{1}$. We will consider cylinder sets of Ω to be the intersection of the usual cylinder sets (for the continued fraction expansion) of $[0, 1)$ with Ω .

We wish to extend the map T to a transformation \tilde{T} on a larger domain $\tilde{\Omega} = \Omega \times \mathcal{M}$ for some finite set \mathcal{M} . For any $(x, M) \in \tilde{\Omega}$, we define

$$\tilde{T}(x, M) = (Tx, f_{a_1(x)}(M)),$$

where $a_1(x) = \lfloor x^{-1} \rfloor$ is the first continued fraction digit of x and the functions $f_a : \mathcal{M} \rightarrow \mathcal{M}$, $a \in \mathbb{N}$, are bijective. Since the second coordinate of $\tilde{T}(x, M)$ only depends on M and the first digit of x , we see that this second coordinate is constant for all x in the same rank 1 cylinder. Given a cylinder set C_s for Ω , we call $C_s \times \{M\}$ (for any $M \in \mathcal{M}$) a cylinder set for $\tilde{\Omega}$. We also have a measure $\tilde{\mu}$ on $\tilde{\Omega}$ that is defined as being the product of the Gauss measure on Ω times the counting measure on \mathcal{M} , normalized by $1/|\mathcal{M}|$ to be a probability measure. By the assumed bijectivity of f , we have that \tilde{T} preserves $\tilde{\mu}$.

For easier readability, we will use (E, M) to denote $E \times \{M\}$ for any measurable set $E \subset \Omega$, with measurability being determined by Lebesgue measure or, equivalently, the Gauss measure.

We adapt our definition of normality on this space. We will say that $(x, M) \in \tilde{\Omega}$ is \tilde{T} -normal with respect to $\tilde{\mu}$, if for any cylinder set (C_s, M') we have

$$\lim_{n \rightarrow \infty} \frac{\#\{0 \leq i < n : \tilde{T}^i(x, M) \in (C_s, M')\}}{n} = \tilde{\mu}(C_s, M').$$

We say \tilde{T} is transitive if for any $M_1, M_2 \in \mathcal{M}$, there exists a proper string s of length n such that

$$T^n(C_s, M_1) = (\Omega, M_2).$$

Theorem 2.1. *If \tilde{T} is transitive, then \tilde{T} is ergodic with respect to $\tilde{\mu}$. Moreover, if x is normal, then for any $M \in \mathcal{M}$, the point (x, M) is \tilde{T} -normal with respect to $\tilde{\mu}$.*

In [10], this result was proved without assuming the bijectivity of the functions f_a . This results in being unable to assume that $\tilde{\mu}$ is an invariant measure and makes the overall proof significantly more difficult.

3. AN OPERATOR-ANALYTIC LEMMA

Let $A = C_{[1]} = [1/2, 1)$. It can be easily calculated that

$$\mu(C_{[1]}) = \mu(A) = \frac{\log(4/3)}{\log 2} \quad \text{and} \quad \mu(C_{[1,1]}) = \mu(T^{-1}A \cap A) = \frac{\log(10/9)}{\log 2}.$$

Moreover, since T is known to be strong mixing, we have that

$$\lim_{n \rightarrow \infty} \mu(T^{-n}A \cap A) = \mu(A)^2 = \left(\frac{\log(4/3)}{\log 2} \right)^2.$$

Lemma 3.1. *We have*

$$(1) \quad \mu(T^{-n}A \cap A) < \mu(T^{-1}A \cap A)$$

for any integer $n \geq 2$.

Proof. We closely follow a process of Wirsing [12] which established the spectral gap in the transfer operator of T , and in turn gave a very precise estimate of

$$|\mu(T^{-n}[0, x)) - \mu([0, x))|$$

as $n \rightarrow \infty$. Through this process, we prove the bound

$$\left| \frac{\mu(A \cap T^{-n}A)}{\mu(A)} - \mu(A) \right| < \mu(A) - \frac{\log(10/9)}{\log(4/3)} = \frac{\log(4/3)}{\log 2} - \frac{\log(10/9)}{\log(4/3)}, \quad (n \geq 2)$$

which implies (1).

To start with, define $m_n, r_n : [0, 1] \rightarrow \mathbb{R}$ by

$$m_n(x) = \frac{\mu(A \cap T^{-n}[0, x))}{\mu(A)} \quad \text{and} \quad r_n(x) = m_n(x) - \mu([0, x)).$$

Then

$$r_n\left(\frac{1}{2}\right) = \frac{\mu(A \cap T^{-n}[0, 1/2))}{\mu(A)} - 1 + 1 - \mu([0, 1/2)) = \mu(A) - \frac{\mu(A \cap T^{-n}A)}{\mu(A)},$$

and so we want to bound $|r_n(1/2)|$. Next, we introduce the transfer operator of T , which is the map $\hat{T} : L^1(\mu) \rightarrow L^1(\mu)$ satisfying

$$\int_B \hat{T}f \, d\mu = \int_{T^{-1}(B)} f \, d\mu, \text{ for all Borel subsets } B \subseteq [0, 1) \text{ and } f \in L^1(\mu),$$

and is given by the formula

$$(2) \quad (\hat{T}f)(x) = \sum_{k=1}^{\infty} \frac{1+x}{(k+x)(k+1+x)} f\left(\frac{1}{k+x}\right), \quad x \in (0, 1).$$

This formula may be extended in the natural way to functions on $[0, 1]$. When extended, \hat{T} is also an operator from $C^1[0, 1]$ to itself. Moreover, if $f = g$ Lebesgue-a.e. then $\hat{T}f = \hat{T}g$ Lebesgue-a.e. We have

$$m_n(x) = \frac{1}{\mu(A)} \int_0^x (\hat{T}^n 1_A)(t) \, d\mu(t) = \frac{1}{\mu(A) \log 2} \int_0^x (\hat{T}^n 1_A)(t) \frac{dt}{1+t},$$

where 1_A is the indicator function of A . Therefore, m'_n exists Lebesgue-a.e. and

$$(1+x)m'_n(x) = \frac{1}{\mu(A) \log 2} (\hat{T}^n 1_A)(x) \quad \text{for Lebesgue-a.e. } x.$$

Now by (2), we clearly have

$$(\hat{T}1_A)(x) = \frac{1}{2+x}$$

if $x \in (0, 1)$. So if we define $f_1(x) = \frac{1}{(2+x)\mu(A)\log 2}$ and $f_n = \hat{T}^{n-1}f_1$, then $f_n = \frac{1}{\mu(A)\log 2}\hat{T}^n 1_A$ Lebesgue-a.e. Since \hat{T} preserves continuity on $[0, 1]$, each f_n is continuous, so we can say that m'_n exists on all of $[0, 1]$, and $f_n(x) = (1+x)m'_n(x)$ for all $x \in [0, 1]$ and $n \in \mathbb{N}$.

Next, we define $g_n(x) = f'_n(x)$, noting that $f_n \in C^1[0, 1]$ for all $n \in \mathbb{N}$. We then have $g_{n+1}(x) = -(Ug_n)(x)$, where U is the operator examined by Wirsing, defined by $U(f') = -(\hat{T}f)'$, and can be shown to be given by

$$(Ug)(x) = \sum_{k=1}^{\infty} \left(\frac{k}{(k+1+x)^2} \int_{1/(k+1+x)}^{1/(k+x)} g(y) dy + \frac{1+x}{(k+x)^3(k+1+x)} g\left(\frac{1}{k+x}\right) \right).$$

The operator U is clearly positive so that $Ug \leq Uf$ whenever $g \leq f$.

We have $f_1(x) = 1/((x+2)\log(4/3))$, and so $g_1(x) = -1/((x+2)^2\log(4/3))$. From the work of Wirsing, $U(-g_1) \leq -\frac{1}{2}g_1$. This can be shown as follows. Let $a(x) = 1/(x+2)^2$, $b(x) = 1/(1+2x)^2$, and $c(x) = -1/(2+4x)$ so that $a \leq b$ on $[0, 1]$ and $c' = b$. For $x \in [0, 1]$, we have

$$\begin{aligned} (Ua)(x) &\leq (Ub)(x) = -(\hat{T}c)'(x) = \frac{d}{dx} \sum_{k=1}^{\infty} \frac{1+x}{(k+x)(k+1+x)} \frac{1}{2+4/(k+x)} \\ &= \frac{1}{2} \frac{d}{dx} \sum_{k=1}^{\infty} \frac{1+x}{(k+1+x)(k+2+x)} = \frac{1}{2} \frac{d}{dx} \sum_{k=1}^{\infty} \left(\frac{1+x}{k+1+x} - \frac{1+x}{k+2+x} \right) \\ &= \frac{1}{2} \frac{d}{dx} \left(\frac{1+x}{2+x} \right) = \frac{1}{2(2+x)^2} = \frac{1}{2} a(x), \end{aligned}$$

implying that $g_2 = U(-g_1) \leq -\frac{1}{2}g_1$, and hence, by iterating this procedure and recalling that $g_{n+1} = -Ug_n$, we get that $|g_n| \leq -\frac{1}{2^{n-1}}g_1$.

Now let $\xi = \log(1+x)$ and $\varrho_n(\xi) = r_n(x)$. Then note that

$$\begin{aligned} \varrho_n''(\xi) &= \frac{d^2}{d\xi^2} r_n(e^\xi - 1) = \frac{d}{d\xi} (e^\xi r'_n(e^\xi - 1)) = e^\xi r'_n(e^\xi - 1) + e^{2\xi} r''_n(e^\xi - 1) \\ &= (1+x)(r'_n(x) + (1+x)r''_n(x)) \\ &= (1+x) \left(m'_n(x) - \frac{1}{(1+x)\log 2} + (1+x) \left(m''_n(x) + \frac{1}{(1+x)^2 \log 2} \right) \right) \\ &= (1+x)(m'_n(x) + (1+x)m''_n(x)) = (1+x) \frac{d}{dx} ((1+x)m'_n(x)) = (1+x)g_n(x). \end{aligned}$$

We have $r_n(0) = r_n(1) = 0$, $\varrho_n(0) = \varrho_n(\log 2) = 0$, and so by the mean value theorem of divided differences,

$$\varrho_n(\xi) = -\xi(\log 2 - \xi) \frac{\varrho_n''(\xi^*)}{2}$$

for some $\xi^* \in [0, \log 2]$ depending on ξ . Letting $\xi = \log(3/2)$ and taking absolute values yields

$$\left| r_n\left(\frac{1}{2}\right) \right| \leq \frac{1}{2} \left(\log \frac{3}{2} \right) \left(\log 2 - \log \frac{3}{2} \right) \|\varrho_n''\|_{\infty} = \frac{1}{2} \left(\log \frac{3}{2} \right) \left(\log \frac{4}{3} \right) \|(1+x)g_n(x)\|_{\infty}$$

$$= \frac{1}{2^n} \left(\log \frac{3}{2} \right) \left(\log \frac{4}{3} \right) \|(1+x)g_1(x)\|_\infty \leq \frac{1}{2^n} \log \frac{3}{2} \left\| \frac{1+x}{(x+2)^2} \right\|_\infty = \frac{1}{2^{n+2}} \log \frac{3}{2}.$$

If $n \geq 2$, this is at most $\frac{1}{16} \log \frac{3}{2} = 0.025341\dots$, which is less than $\frac{\log(4/3)}{\log 2} - \frac{\log(10/9)}{\log(4/3)} = 0.048798\dots$. This completes the proof of Lemma 3.1. \square

4. PROOF OF THEOREM 1.1

Without loss of generality, it suffices to prove the theorem if $1 \leq k \leq m$.

Consider the augmented system \tilde{T} on $\tilde{\Omega}$ given by $\mathcal{M} = \{1, 2, \dots, m\}$ and $f_a(k) = k + 1 \pmod{m}$ for all $a \in \mathbb{N}$. In particular, we always have that

$$\tilde{T}^i(x, j) = (T^i x, j + i \pmod{m}).$$

Also, it is clear that this is transitive: for any rank n cylinder, we have that $\tilde{T}^n(C_s, j_1) = (\Omega, j_1 + n \pmod{m})$. Therefore Theorem 2.1 applies.

Let $x = [a_1, a_2, a_3, \dots]$ be CF-normal, and let $y = [a_k, a_{m+k}, a_{2m+k}, \dots]$. Consider the string $s = [1, 1]$. We want to show that the limiting frequency of s in the digits of y does not equal $\mu(C_s)$.

Borrowing our notation from the last section, we let $A = C_{[1]}$ and we will now denote $A \cap T^{-n}A$ by E_n , so that $C_s = E_1$.

We have that $T^i y \in E_1$ if and only if $T^{mi+k-1}x \in E_m$. Note that $(x, 1)$ is normal with respect to \tilde{T} by Theorem 2.1. Thus we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\#\{0 \leq i \leq n : T^i y \in C_s\}}{n} &= \lim_{n \rightarrow \infty} \frac{\#\{0 \leq i \leq n : T^{mi+k-1}x \in E_m\}}{n} \\ &= \lim_{n \rightarrow \infty} \frac{\#\{0 \leq i \leq mn : \tilde{T}^i(x, 1) \in (E_m, k)\}}{n} \\ &= m \cdot \lim_{n \rightarrow \infty} \frac{\#\{0 \leq i \leq mn : \tilde{T}^i(x, 1) \in (E_m, k)\}}{mn} \\ &= m \cdot \tilde{\mu}(E_m, k) = m \cdot \frac{\mu(E_m)}{m} = \mu(E_m). \end{aligned}$$

By Lemma 3.1, we have that $\mu(E_m) < \mu(C_s)$, which proves the theorem.

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